Question one (13 marks)

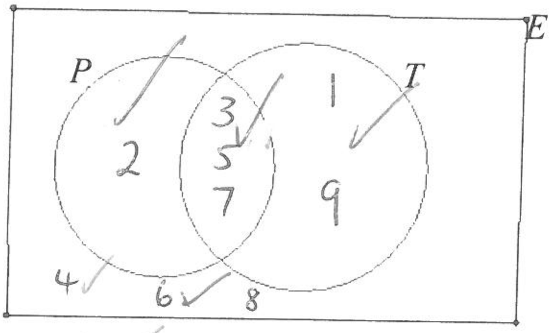
1.1

a) Insert numbers into the Venn diagram below so that :

E is the set of natural numbers less than 10. $E = \{1, 2, 3, 4, \dots, 9\}$

P is the set of prime numbers in E. $P = \{2, 3, 5, 7\}$

T is the set of odd numbers in E. $T = \{1, 3, 5, 7, 9\}$ (5)

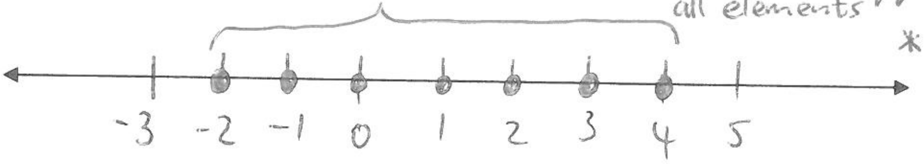


b) $P \cup T = \{1, 2, 3, 5, 7, 9\}$ ✓

c) Determine $n(P \cup T)$. 6 ✓ (2)

1.2 Illustrate on the number line: all integers between - 3 and 5. (2)

all elements ✓✓



*if one extra or one missing then ✓.

1.3 Prove that $0.\dot{8}$ is a rational number. (4)

Proof: Let $x = 0.\dot{8}$

then $x = 0.888\ldots$ ½ $\ldots(1)$

$10x = 8.888\ldots$ ½ $\ldots(2)$

$(2)-(1)$: $9x = 8$

$x = \frac{8}{9}$ which is of the form $\frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$

$\therefore$ $0.\dot{8}$ is a rational number.

Question two (16 marks)

2. Solve for x:

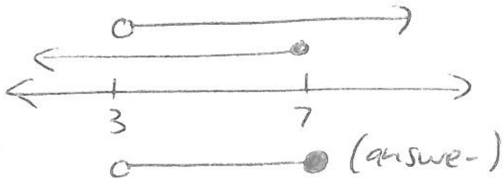
a) $3 < 3x - 6 \leq 1 + 2x$ and write your answer in set builder notation. (5)

$3 < 3x - 6$ and $3x - 6 \leq 1 + 2x$

$9 < 3x$ $\qquad$ $x \leq 7 \ldots (2)$

$3 < x$

$x > 3 \ldots (1)$



$\{x \in \mathbb{R} : 3 < x \leq 7\}$

b) $\left|5 + \frac{x}{2}\right| = 9$ (3)

½ $\qquad$ ½

$5 + \frac{x}{2} = 9$ OR $5 + \frac{x}{2} = -9$

$\frac{x}{2} = 4$ $\qquad$ $\frac{x}{2} = -14$

$x = 8$ $\qquad$ $x = -28$

c) $2|3x+4|-1 \geq 5$ (4)

$|3x+4| \geq 3$

$3x+4 \geq 3$ ✓ or $3x+4 \leq -3$ ✓

$3x \geq -1$ $3x \leq -7$

$x \geq -\frac{1}{3}$ ✓ $x \leq \frac{-7}{3}$

$x \leq -2\frac{1}{3}$ ✓

d) $x^2 - 6x \geq 0.$ (4)

$x(x-6) \geq 0$

Critical values: 0, 6 ✓

	$x<0$	0	$0<x<6$	6	$x>6$
$x(x-6)$	+	0	−	0	+

✓

$x \leq 0$ ✓ or $x \geq 6$ ✓

<u>**Question three (20 marks)**</u>

3.1 a) Simplify $\dfrac{9^{2n+1} \times 81^{n-2}}{\sqrt[3]{27^n}}$. (3)

$= \dfrac{(3^2)^{2n+1} \; \frac{1}{2} \; (3^4)^{n-2}}{(27)^{\frac{n}{3}}}$

$= \dfrac{3^{4n+2} \checkmark \frac{1}{2} \times 3^{4n-8} \checkmark \frac{1}{2}}{3^n \; \frac{1}{2}}$

$= \dfrac{3^{8n-6}}{3^n} \; \frac{1}{2}$

$= 3^{7n-6}$ ✓

b) Solve for x: $25^x - 2 \times 5^{x+1} = -25$ (4)

$5^{2x} - 2 \times 5^x \times 5 = -25$ $\frac{1}{2}$

$(5^x)^2 - 10 \times 5^x = -25$ $\frac{1}{2}$ $\frac{1}{2}$

Let $5^x = k$ then $k^2 - 10k + 25 = 0$ $\frac{1}{2}$

$(k-5)(k-5) = 0$

$k = 5$

$5^x = 5$

$x = 1$

3.2 Simplify without the use of calculators:

a) $2 \log 2 + 2 \log 5 = \log 4 + \log 25$ $\frac{1}{2}$ (2)

$= \log 100$ $\frac{1}{2}$

$= 2 \log 10$

$= 2$

b) $\log_9 27 - \log_2 \sqrt{8} = \dfrac{3\log 3}{2\log 3}$ $\frac{1}{2}$ $- \dfrac{\frac{1}{2}\log_2 8}{1}$ $\frac{1}{2}$ (3)

$= \frac{3}{2}$ $\frac{1}{2}$ $- \frac{3}{2}\log_2 2$ $\frac{1}{2}$ $= \frac{3}{2} - \frac{3}{2} = 0$

3.3 a) Solve for x: $\log_8 x + \log_8 (x+2) = 1$ (4)

$\log_8 (x^2 + 2x) = 1$ NB [illegible] of logs has been used here!

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$ $\frac{1}{2}$ factors

$x = -4$ $\frac{1}{2}$ or $x = 2$

n/a

b) Solve for $x : \log_m x = \dfrac{4}{\log_3 m}$ (4)

$$\frac{\log_3 x}{\log_3 m} = \frac{4}{\log_3 m} \quad \therefore \quad \log_3 x = 4 \quad (m \neq 1)$$

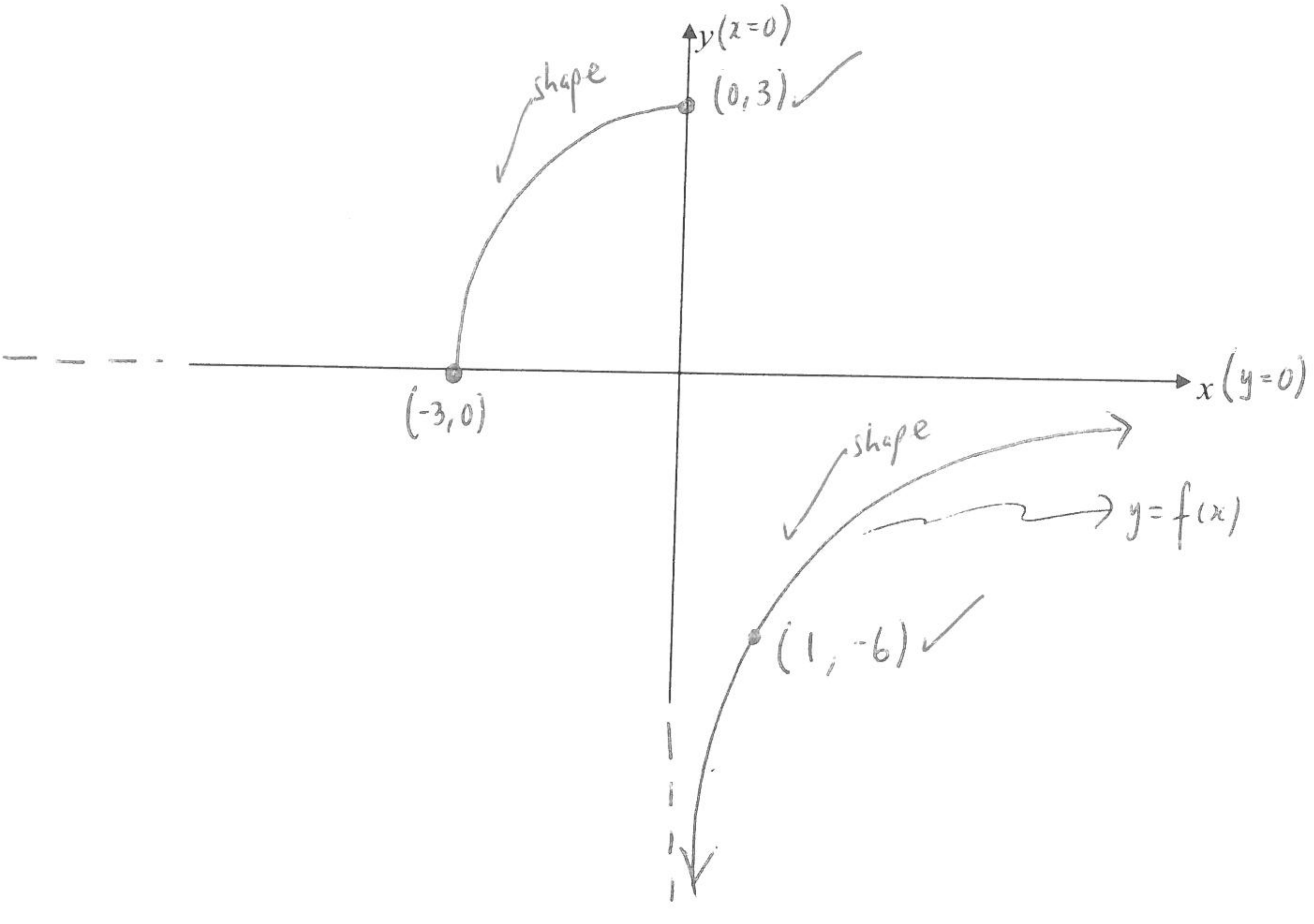
$$x = 3^4$$

$$x = 81$$

Question four (13 marks)

4.1 Given the function f: $f(x) = \begin{cases} \sqrt{9 - x^2} & , \ -3 \le x \le 0 \\ -\dfrac{6}{x} & , \ x > 0 \end{cases}$ (4)

Sketch the graph of f on the set of axes below.



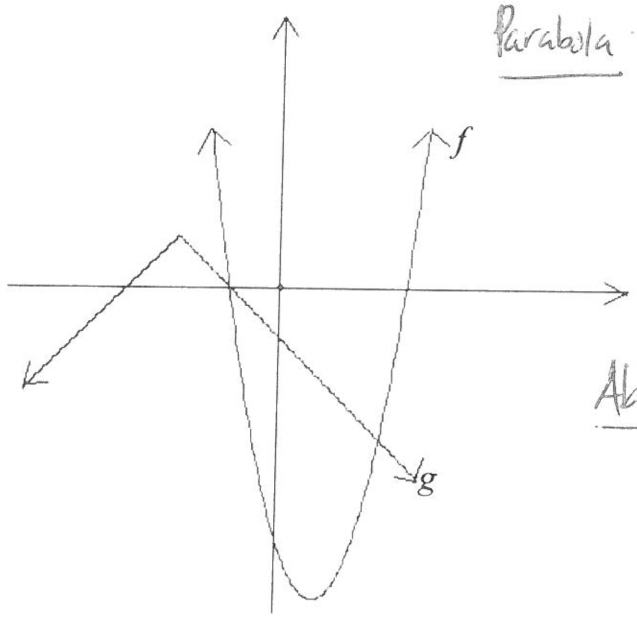
4.2 Determine the centre of the circle and radius given the equation $x^2 + y^2 + 4y = 3 + 6x$. (4)

$$x^2 - 6x + (-3)^2 + y^2 + 4y + (2)^2 = 3 + 9 + 4$$

$$(x-3)^2 + (y+2)^2 = 16$$

$\odot(3, -2)$ with radius $= 4$ units

4.3 The graphs of $f(x) = ax^2 + bx + c$ and $g(x) = k|x - l| + n$ are sketched below. The graph of f has x-intercepts of -1 and 3 and cuts the y-axis at -3. The graph of g has salient point (-3, 5) and $g(1) = -2$. Determine the value(s) of a, b, c, k, l and n. (5)



Parabola

$$f(x) = a(x+1)(x-3)$$

$$-3 = a(1)(-3)$$

$$a = 1$$

$$f(x) = (x+1)(x-3) = x^2 - 2x - 3$$

$\therefore$ $a = 1$, $b = -2$ and $c = -3$

Absolute Value:

$$g(x) = k|x - l| + n$$

$$g(x) = k|x+3| + 5$$

$$-2 = k|1+3| + 5$$

$$-2 = 4k + 5$$

$$4k = -7$$

$$k = \frac{-7}{4}$$

$$l = -3$$

$$n = 5$$

Question five (32 marks)

5.1 Use first principles of differentiation to find $f'(x)$ of the following function:

$$f(x) = 2x^2 - 3$$ (4)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^2 - 3 - (2x^2 - 3)}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(4x + 2h)}{h}$$

$$= \lim_{h \to 0} (4x + 2h)$$

$$= 4x \quad \therefore f'(x) = 4x$$

5.2 Use the rules of differentiation to determine:

a) $\dfrac{dy}{dx}$ if $y = (x^2 - 2x)^2$ (3)

$$y = x^4 - 4x^3 + 4x^2$$

$$\frac{dy}{dx} = 4x^3 - 12x^2 + 8x$$

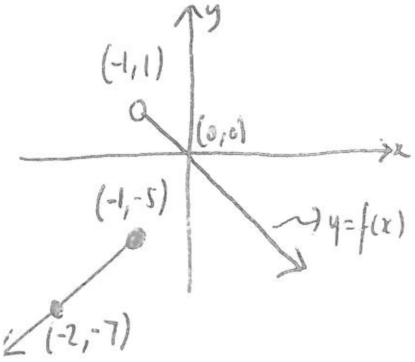
b) $D_t\left[\dfrac{t^{-1} + 4\sqrt{t}}{t^3}\right] = D_t\left[t^{-4} + 4t^{-\frac{5}{2}}\right]$ (4)

$$= -4t^{-5} - 10t^{-\frac{7}{2}}$$

$$= \frac{-4}{t^5} - \frac{10}{\sqrt{t^7}}$$

5.3 Consider the following function: $f(x) = \begin{cases} 2x-3, & x \le -1 \\ -x, & x > -1 \end{cases}$

Determine $\lim\limits_{x \to -1} f(x)$ (4)



$\lim\limits_{x \to -1^-} f(x) = -5$ and $\lim\limits_{x \to -1^+} f(x) = 1$

$\therefore \lim\limits_{x \to -1} f(x)$ does not exist.

5.4 Given functions f and g where $f(x) = x^3 - 3x - 2$ and $g(x) = 2x - 2$.

* take wording from supp.

a) Sketch the graphs of f and g on the set of axes below. (8)

For $y = f(x)$: (Clearly showing all ...)

y-int $(0,-2)$

$f'(x) = 3x^2 - 3 = 0$

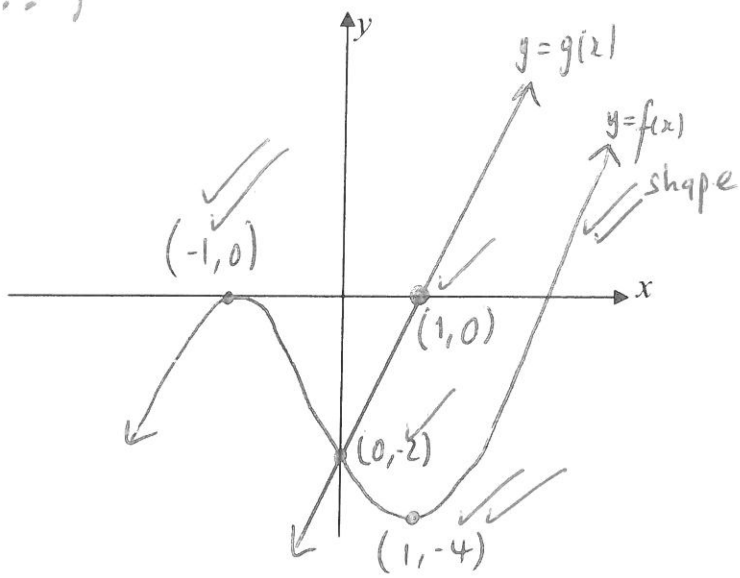
$x^2 = 1$

$x = \pm 1$

$x = 1$ or $x = -1$

$y = -4$ $\quad$ $y = 0$

		-1		1	
$f'(x)$	+	0	−	0	+
		Max. occurs		min. occurs	



b) Find the equation of the tangent to f where $x = 2$. (4)

$m_{tangent} = f'(2) = 3(2)^2 - 3 = 9 \quad \therefore y = 9x + c$

Point of tangency $(2, f(2))$ i.e. $(2, 0)$

$\therefore y = 9x + c$ becomes $0 = 18 + c$

$c = -18$

required tangent is $y = 9x - 18$

c) Find $(g \circ f)(x)$. (2)

$(g \circ f)(x) = g(f(x))$ ✓
$= g(x^3 - 3x - 2)$
$= 2(x^3 - 3x - 2) - 2$
$= 2x^3 - 6x - 6$ ✓

d) Write down the turning points of $m(x) = -x^3 + 3x + 2$ (3)

$m(x) = -(x^3 - 3x - 2)$
$= -f(x)$ ✓

required turning points is $(-1, 0)$ ✓ and $(1, 4)$ ✓.

Question six (43 marks)

6.1 Simplify: $\dfrac{\cos 330° \tan 150° \sin 12°}{\cos(-258°)}$ (5)

$= \dfrac{\cos 30° \times -\tan 30° \times \sin 12°}{\cos 258°}$

$= \dfrac{\frac{\sqrt{3}}{2} \times -\frac{1}{\sqrt{3}} \times \sin 12°}{\cos(270° - 12°)}$

$= \dfrac{-\frac{1}{2} \times \sin 12°}{-\sin 12°}$

$= \dfrac{1}{2}$ ✓

6.2 If $\cos 20° = p$, (without using a calculator) write down in terms of p:

a) $\cos 380°$ (3)

$= \cos(20° + 1 \times 360°)$

$= \cos 20°$ ✓

$= p$ ✓✓

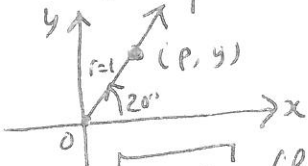
b) $\tan 200° = \tan(180° + 20°)$ (3)

$= \tan 20°$ ✓

$= \dfrac{y}{x}$

$= \dfrac{\sqrt{1-p^2}}{p}$ ✓ ✓

but $\cos 20° = p = \dfrac{p}{1}$



$y = \sqrt{1-p^2}$ (Pythagoras)

6.3 Simplify $$\frac{\sin(2\pi - \theta) - \cos\left(\frac{\pi}{2} + \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right) - \sin(-\theta)}$$ (4)

$$= \frac{-\sin\theta ✓ - (-\sin\theta) ✓}{\sin\theta ✓ \tfrac{1}{2} + \sin\theta ✓ \tfrac{1}{2}}$$

$$= \frac{-\sin\theta + \sin\theta}{2\sin\theta}$$

$= 0$ ✓

6.4 ai) Prove ; $\dfrac{1-\tan\theta}{1+\tan\theta} = \dfrac{\cos 2\theta}{1+\sin 2\theta}$ (5)

Proof: $\text{RHS} = \dfrac{\cos^2\theta - \sin^2\theta \checkmark}{1+2\sin\theta\cos\theta \checkmark}$

$= \dfrac{(\cos\theta - \sin\theta)(\cos\theta+\sin\theta)}{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta}$

$= \dfrac{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta) \checkmark \frac{1}{2} \text{ factors}}{(\cos\theta+\sin\theta)(\cos\theta+\sin\theta) \checkmark \frac{1}{2} \text{ factors}}$

$= \dfrac{(\cos\theta - \sin\theta)\frac{1}{\cos\theta}}{(\cos\theta + \sin\theta)\frac{1}{\cos\theta}}$

$= \dfrac{1 - \frac{\sin\theta}{\cos\theta}}{1 + \frac{\sin\theta}{\cos\theta}}$

$= \dfrac{1-\tan\theta \checkmark}{1+\tan\theta \checkmark}$

$= \text{LHS}$ $\therefore \dfrac{1-\tan\theta}{1+\tan\theta} = \dfrac{\cos 2\theta}{1+\sin 2\theta}$

aii) Hence determine (without the use of a calculator) $\dfrac{1-\tan 22.5°}{1+\tan 22.5°}$. (4)

Set $\theta = 22.5°$ in the above identity

$\therefore \dfrac{1-\tan 22.5°}{1+\tan 22.5°} = \dfrac{\cos 45° \checkmark}{1+\sin 45° \checkmark} = \dfrac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \dfrac{1}{\sqrt{2}} \div \left(\dfrac{\sqrt{2}+1}{\sqrt{2}}\right)$

$= \dfrac{1 \checkmark}{\sqrt{2}+1 \checkmark} \times \dfrac{\sqrt{2}-1}{\sqrt{2}-1}$

$= \dfrac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$

aiii) Find the value(s) for $\theta \in [0, \ 2\pi]$ for which the identity in (ai) is undefined. (3)

$\theta \neq (2k+1) \times \frac{\pi}{2}, \ k \in \mathbb{Z}$ OR $\tan\theta \neq -1$ OR $\sin 2\theta \neq -1$

$k=0: \ \theta \neq \frac{\pi}{2}$
$k=1: \ \theta \neq \frac{3\pi}{2}$ $\checkmark$

OR $\theta \neq \frac{3\pi}{4}$, $\theta \neq \frac{7\pi}{4}$ $\checkmark$

$2\theta \neq \frac{3\pi}{2}$, $\theta \neq \frac{3\pi}{4}$ $\checkmark$

6.5 Determine the general solution of θ:

$\cos 2\theta = -5\cos\theta + 2$, where θ is an angle measured in degrees. (5)

$2\cos^2\theta - 1 = -5\cos\theta + 2$

$2\cos^2\theta + 5\cos\theta - 3 = 0$

$(2\cos\theta - 1)(\cos\theta + 3) = 0$ factors

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -3$ n/a (amplitude of $y = \cos\theta$ is 1 unit, reject

ref ∠ = 60°

$\theta = 60^\circ + k \times 360^\circ, k \in \mathbb{Z}$

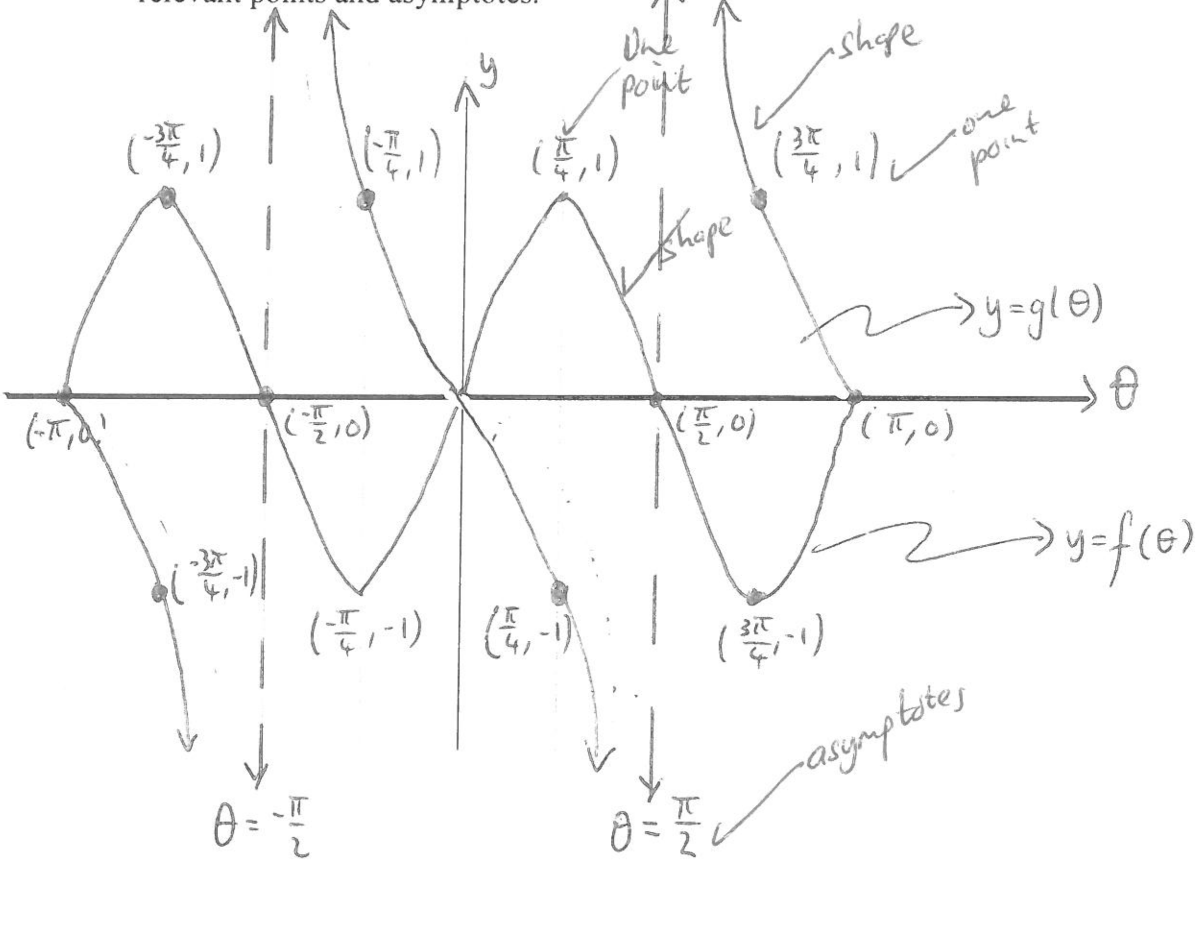
or $\theta = 300^\circ + k \times 360^\circ, k \in \mathbb{Z}$

6.6 Consider the following functions:

$f(\theta) = \sin 2\theta \quad , \quad \theta \in [-\pi, \pi]$ Period $= \frac{2\pi}{2} = \pi$

$g(\theta) = -\tan\theta, \ \theta \in [-\pi, \pi]$ Period $= \frac{\pi}{1} = \pi$

a) Sketch graphs of f and g on the set of axes below, clearly labelling all relevant points and asymptotes. (5)



b) Write down the period of f and the amplitude of g. (2)

Period(f) = π, g has no amplitude

c) Write down the range of f. (2)

Range(f) = $y \in [-1, 1]$

d) Explain how you would use the graph of $y = g(\theta)$ to obtain the graph of
$y = 1 - \tan\theta, \quad \theta \in [-\pi, \pi]$. (2)

Shift the graph of $y = g(\theta)$ 1 unit up (vertically).

Question seven (13 marks)

7.1 A fish tank (open at the top) is completely full with water and is in the shape of a right triangular prism with triangle ABC being equilateral as shown in Fig.1.

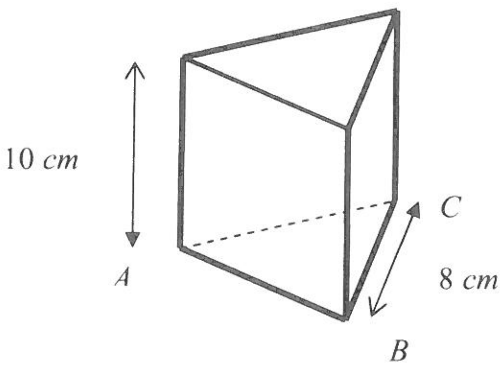


Fig. 1

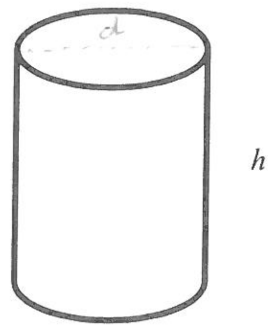


Fig. 2

a) Determine the total surface area of the fish tank. (4)

TSA of fish tank = Area $\triangle ABC$ + $3(8 \times 10)$ cm^2
$= \left(\frac{1}{2} \times 8 \times \sqrt{64-16} + 240\right)$ cm^2
$= (4\sqrt{48} + 240)$ cm^2
$= (16\sqrt{3} + 240)$ cm^2

b) If all the water is transferred into the right cylinder with diameter d and height h as shown in Fig.2, determine the height of the water. (4)

Volume of fish tank = Volume of cylinder ✓

$$4\sqrt{48} \times 10 = \pi r^2 h ✓$$

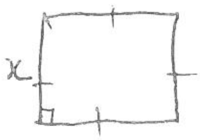
$$h = \frac{40\sqrt{48}}{\pi r^2} ✓$$

$$= \frac{40\sqrt{48}}{\pi\left(\frac{d}{2}\right)^2}$$

$$= \frac{40\sqrt{48}}{\frac{\pi d^2}{4}}$$

$$= \frac{160\sqrt{48}}{\pi d^2} \text{ cm}$$

7.2 A piece of wire 18 cm long is cut into two pieces. One piece is bent to form a square of side x cm and the other is bent to form a circle.
Determine the value(s) of x so that the total area of the square and circle is a minimum. (5)



Amount of wire used for square

$= (4x)$ ✓ $\frac{1}{2}$ cm.

Amount remaining for circle $= (18 - 4x)$ cm ✓ $\frac{1}{2}$

$\Rightarrow 2\pi r = 18 - 4x$

$$r = \frac{18-4x}{2\pi} = \frac{9-2x}{\pi} \text{ cm} ✓$$

$$\text{Area} = A = x^2 + \pi\left(\frac{9-2x}{\pi}\right)^2 ✓$$

$$A = x^2 + \frac{\pi}{1}\left(\frac{81 - 36x + 4x^2}{\pi^2}\right)$$

$$= x^2 + \frac{81}{\pi} - \frac{36}{\pi}x + \frac{4}{\pi}x^2$$

$$= \left(1 + \frac{4}{\pi}\right)x^2 + \frac{81}{\pi} - \frac{36}{\pi}x$$

$$\frac{dA}{dx} = 2\left(1 + \frac{4}{\pi}\right)x - \frac{36}{\pi} = 0 \quad \text{(for max./min)}$$

$$x = \frac{36}{\pi} \div \left(\frac{2\pi + 8}{\pi}\right) = \frac{36}{2\pi + 8} = \frac{18}{\pi + 4} ✓$$

verify min.

		$\frac{18}{\pi+4}$	
$\frac{dA}{dx}$	$-$	0	$+$ ✓
	\	min.	/

$\therefore$ for ~~loc~~ minimum area, $x = \frac{18}{\pi+4}$.