GRADE 12 EXAMINATION
NOVEMBER 2009

ADVANCED MATHEMATICS

Time: 2½ hours 120 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 16 pages, and a 4-page Formula Sheet. Please check that your question paper is complete.
2. Answer ALL the questions set on FOUR of the modules you have studied.
3. Non-programmable calculators may be used, unless otherwise indicated.
4. All necessary calculations must be clearly shown and writing should be legible.
5. Diagrams have not been drawn to scale.

MODULE 1 INTEGRATION

QUESTION 1

Given that $\int_1^2 f(x)dx = 10$ and $f(x) = f(-x)$, find the value of:

1.1 $\int_1^2 2f(x)dx$ (1)

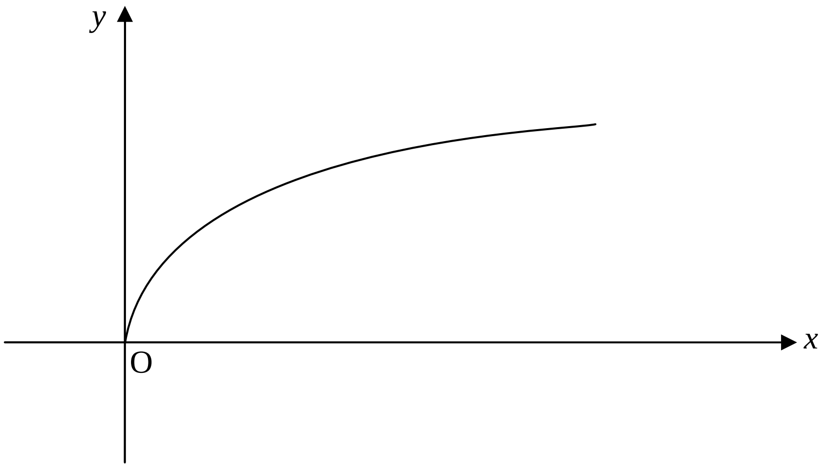
1.2 $\int_{-2}^{-1} f(x)dx$ (1)

1.3 $\int_2^1 [f(x)+3]dx$ (4)

[6]

QUESTION 2

A sketch of the graph of $y = \sqrt{x}$ is given below.



By considering the area between the curve $y = \sqrt{x}$ and the x-axis between $x = 0$ and $x = 25$ as a sum of rectangular strips of width 1 unit, show that:

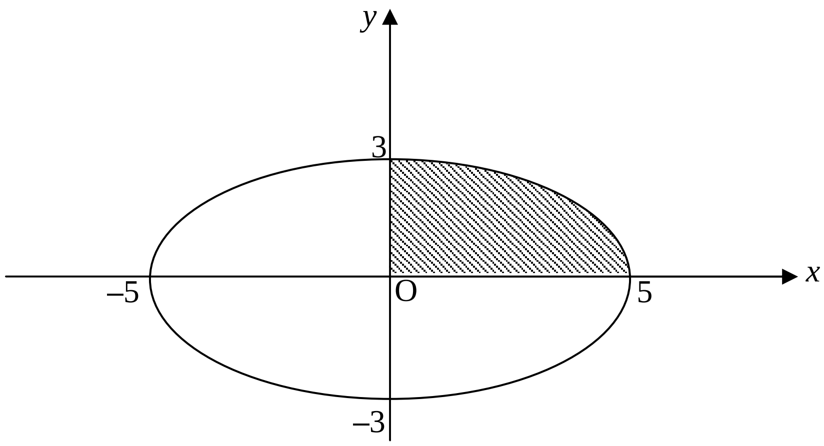
$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \ldots\ldots\ldots\ldots\ldots \sqrt{25} > \frac{250}{3}$$

[5]

QUESTION 3

The equation of the ellipse below is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

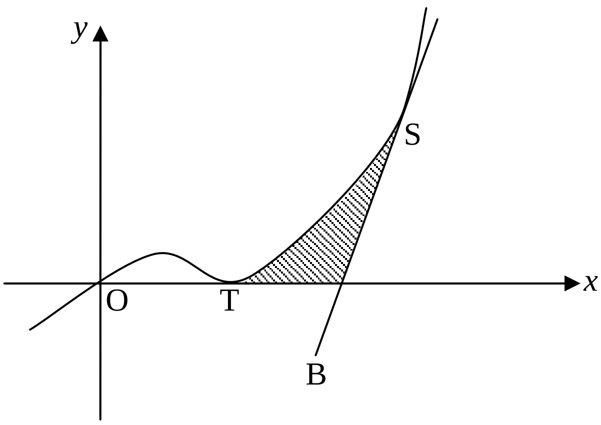


Calculate the volume generated by rotating the shaded region through 360^{o} about the x axis.

[7]

QUESTION 4

The graph of $y = x(x-1)^2$ touches the *x*-axis at T.
S is the point (2; 2) on the curve and SB is the tangent to the curve at S.



4.1 Show that the equation of the tangent at S is $y = 5x - 8$ (4)

4.2 Calculate the area of the shaded region. (8)

[12]

30 marks

MODULE 2 DIFFERENTIATION

QUESTION 1

Find:

1.1 $\lim_{\theta \to 0} \frac{\sin 3\theta}{2\theta}$ (2)

1.2 $f'(\theta)$ if $f(\theta) = \sin\theta \tan\theta$ (2)

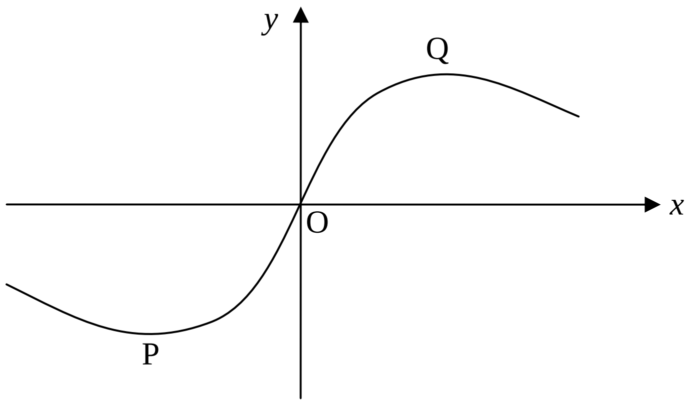
1.3 $g'(\theta)$ if $g(\theta) = \sin(\tan\theta)$ (2)

1.4 $\frac{dy}{dx}$ if $y = \sqrt{2x^4 - 3x - 1}$, leaving your answer unsimplified. (3)

[9]

QUESTION 2

A sketch of the graph of $y = \frac{4x}{x^2 + 4}$ appears below.



2.1 Determine the co-ordinates of the turning points P and Q. (6)

2.2 If line QO is drawn, where O is the origin, calculate the angle, in radians, that QO makes with the positive x-axis. (Answer correct to 3 decimal places.) (2)

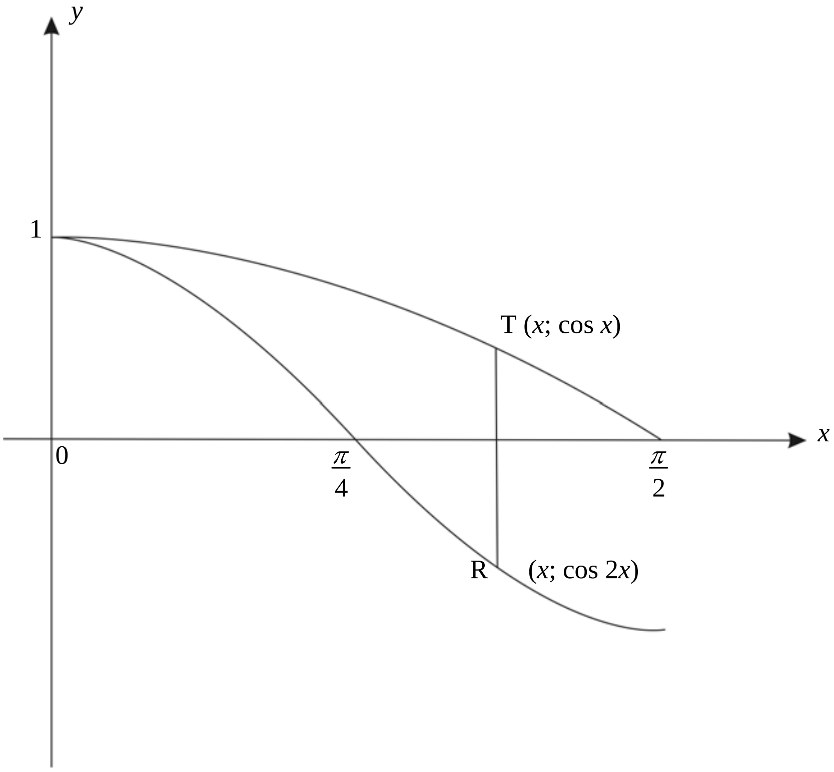
[8]

QUESTION 3

The graphs of f and g have equations $y = \cos x$ and $y = \cos 2x$ respectively, for the interval

$$0 \leq x \leq \frac{\pi}{2}$$

T and R are variable points on f and g respectively such that line TR is parallel to the y-axis.



3.1 Write down an expression in x for the length of line TR. (1)

3.2 Find the value of x in the interval $0 \leq x \leq \frac{\pi}{2}$ for which TR has its maximum length. (6)

[7]

QUESTION 4

The photograph of the Moses Mabhida stadium in Durban shows the grand Centre Arch which spans the entire stadium. (See Figure 1)
A cable-car (or funicular) runs on a rail which is mounted on the arch. This can transport 25 passengers at a time to the viewing platform at the highest point on the arch.

Figure 1

Figure 2 is a diagram of the arch in the shape of a parabola with equation

$$y = -\frac{3}{875}x^2 + \frac{6}{5}x$$

The funicular moves along the rail towards the viewing platform so that the horizontal distance from O is changing at a steady rate of 0,95 metres per second, i.e. the x value is changing at a rate of 0,95 metres per second.
Calculate the rate of change of the height of the funicular above ground when it has moved a horizontal distance of 100 metres from O.

Figure 2

Starting Station O

[6]

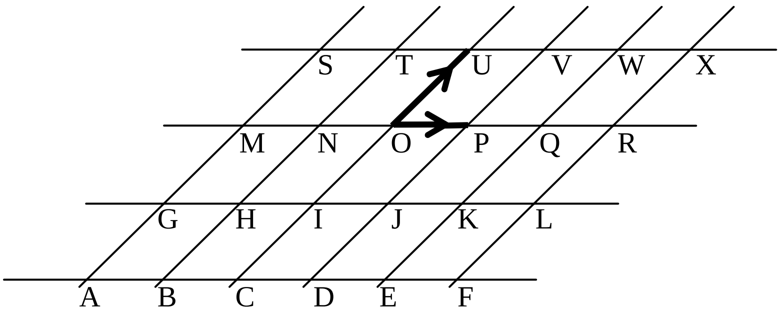
30 marks

MODULE 3 VECTORS

QUESTION 1

The grid in Figure 1 is made up of congruent parallelograms.

Figure 1



If $\underline{OP} = \underline{p}$ and $\underline{OU} = \underline{u}$, express the following vectors in terms of $\underline{p}$ and $\underline{u}$:

1.1 $\underline{JX}$ (2)

1.2 $\underline{SE}$ (2)

[4]

QUESTION 2

2.1 Find a vector equation for line P_1 passing through M(–5; 3; 4) and N(–2; 9; 1). (2)

2.2 The line P_1 meets the *x*-*z* plane at R. Find the co-ordinates of R. (3)

2.3 Give a vector equation for the plane MNR. (2)

[7]

QUESTION 3

3.1 Find the Cartesian equation of plane π_1 which passes through T(1; 3; –2),

given that $\underline{n} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ is normal to π_1. (2)

3.2 The plane π_2 has equation $x + 2y - 2z - 9 = 0$.

(a) Investigate whether or not π_1 and π_2 are parallel planes. (2)

(b) If the perpendicular distance from K(1; k; 1) to the plane π_2 is $\frac{8}{3}$ units, find the value(s) of k. (5)

3.3 Given $\underline{u} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$, find a vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ which is perpendicular to both $\underline{u}$ and $\underline{v}$. (3)

[12]

QUESTION 4

A pair of lines in 2-space have equations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + m\begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}$$

4.1 Prove that the two lines intersect at a point P with co-ordinates $(\cos 2\beta; \sin 2\beta)$. (5)

4.2 Find the distance from the origin to point P for all values of β. (2)

[7]

30 marks

MODULE 4 MATRICES

QUESTION 1

1.1 Given matrix S = $\begin{pmatrix} \cos\alpha & 1 \\ 1 & 2\sin\alpha \end{pmatrix}$

For which values of α in the interval [0^o; 180^o] is S a singular matrix, i.e. not invertible? (4)

1.2 If B = (b_{ij}) is a 4 × 4 matrix, describe the position of the elements for which

(a) **i** = **j** (1)

(b) **j** = 3 (1)

[6]

QUESTION 2

2.1 If M = $\begin{pmatrix} 2 & 3 \\ a & 4 \end{pmatrix}$, solve for a,

given that $M - M^2 = 7\mathbf{I} + \begin{pmatrix} 0 & -15 \\ 15 & -10 \end{pmatrix}$, where **I** is the 2 × 2 unit matrix. (4)

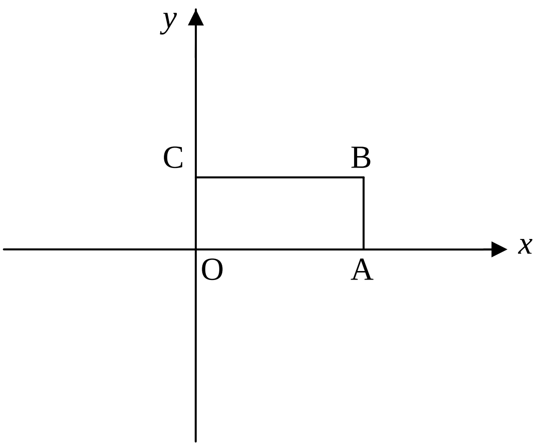
2.2 Given that **A** is a 3 × 3 invertible matrix such that $\mathbf{A}^2 = \mathbf{A} - \mathbf{I}$, show that $\mathbf{A}^4 = -\mathbf{A}$. (4)

[8]

QUESTION 3

3.1 Show that the transformation with matrix $T = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ maps every point on the x-y plane onto the same straight line, and give the equation of the line. (3)

3.2 Rectangle OABC is given below.



(a) Draw a similar pair of axes in your answer book, and on that pair of axes draw rectangle OA′B′C′ which is a reflection of rectangle OABC in the x-axis. (1)

(b) On another pair of axes, draw rectangle OA″B″C″, which is the result of rotating rectangle OA′B′C′ anticlockwise through 90^{o}. (1)

(c) Find the matrix of the composite transformation "reflection in the x-axis followed by an anti-clockwise rotation through 90°". (3)

[8]

QUESTION 4

Use Row Reduction to find the values of p for which the equations

$$\begin{aligned} x + 2y + 3z &= 1 \\ 5x + y + 3z &= p \\ 3x + 9y + 13z &= p^2 \end{aligned}$$

have an infinite number of solutions.

[8]

30 marks

MODULE 5 NUMERICAL METHODS

QUESTION 1

1.1 Investigate whether an error of 2 m in 10 km is more or less than an error of 0,2 cm in 2 m. (3)

1.2 The base and height of a triangle are measured as 16 cm and 23 cm respectively. The possible error in these measurements is $\pm 0,2$ cm.
Find two values between which the area of the triangle must lie. (4)

[7]

QUESTION 2

2.1 Given $f(x) = 2^x - 3$, show graphically that $f(x) = 0$ has only one real root $x = \alpha$. (3)

2.2 Give the integer k such that $k < \alpha < k + 1$. (2)

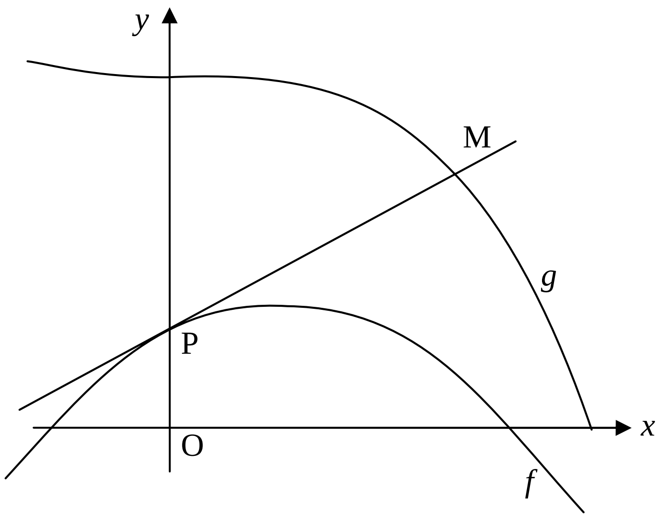
2.3 Now apply the method of False Position (Regula Falsi) to find an approximate solution $x = \overline{\alpha}$ to the equation $f(x) = 0$ such that $| f(\overline{\alpha}) | < 3 \times 10^{-2}$. (6)

[11]

QUESTION 3

The equation of curve f is $y = -x^2 + 2x + 3$. The equation of curve g is $y = -\frac{1}{4}x^3 + 12$.
The tangent to the curve f at point P intersects the curve g at M. (See Figure 1 below)

Figure 1



3.1 Find the equation of the tangent PM. (3)

3.2 Prove that the x value at M lies in the interval [2; 3] (3)

3.3 Use the Newton-Raphson method to find the x value at M, correct to 5 decimal places. (6)

[12]

30 marks

MODULE 6 LOGIC AND BOOLEAN ALGEBRA

QUESTION 1

1.1 Prove by means of a truth table that the statement

$(p \Rightarrow q) \bullet (p \Rightarrow \overline{q}) \Rightarrow \overline{p}$ is a tautology. (5)

1.2 Explain whether the following argument is valid:

p is sufficient for $(q + r)$
p
$\therefore \bar{r} \Rightarrow q$ (3)

1.3 *R(x)* means '*x* is a real number'
M(x) means '*x* has a multiplicative inverse'

Translate each one of the following statements into simple English:
Also say which one is false:

(a) $(\forall x)[R(x) \Rightarrow M(x)]$

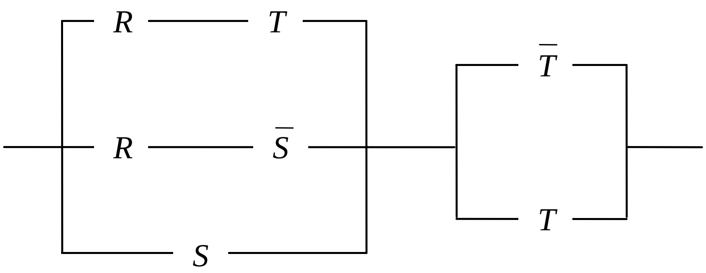
(b) $(\exists x)[[R(x) \bullet \overline{M(x)}]$ (3)

[11]

QUESTION 2

2.1 Let x be an element of a Boolean Algebra.
Prove the theorem $x + x = x$.
Also state the dual of this theorem. (5)

2.2 The sketch represents an electrical network.
Write down and simplify the algebraic representation of this network



(4)

[9]

QUESTION 3

3.1 Prove by mathematical induction that:

$$8 + 15 + 22 + ... + (7n + 1) = \frac{n(7n + 9)}{2}, \text{ for } n \text{ a positive integer.}$$ (6)

3.2 Prove by the method of contradiction that:

If x and y are positive and unequal real numbers, then

$$\sqrt{xy} \leq \frac{x + y}{2}$$ (4)

[10]

30 marks

MODULE 7 PROBABILITY, PERMUTATIONS AND COMBINATIONS

QUESTION 1

1.1 In how many different ways can the letters of the word DIFFERENT be arranged in:

(a) a row? (2)

(b) a row with the first and last letter being the same? (2)

(c) a circle? (2)

(d) a circle with no two adjacent letters the same? (3)

1.2 How many diagonals does a convex polygon with 12 vertices have? (3)

[12]

QUESTION 2

2.1 In how many ways can a group of 4 be selected from 10? (2)

2.2 In how many ways can a group of 10 people be divided into two groups, one with four people and the other with six? (1)

2.3 In how many ways can 10 people be divided into two groups so that Kim will be in the smaller group, but not alone in that group? (3)

[6]

QUESTION 3

In a box of 12 calculators, 3 are faulty.
A random sample of four calculators is drawn from the box.
What is the probability that the sample contains 1 faulty calculator?

[3]

QUESTION 4

A coin is biased so that when tossed it will fall H (heads) uppermost 60% of the time.
The coin is thrown 6 times.
What is the probability that there will be at most 4 H's (heads)?

[4]

QUESTION 5

5.1 If A and B are mutually exclusive events, what is the value of $P(A|B)$? (1)

5.2 Let E and F be events of a sample space S.
It is given that $P(E) = 0{,}2$
$P(F) = 0{,}7$
and $P(\overline{E \cap F}) = 0{,}86$

(a) Are E and F independent? Explain. (2)

(b) Calculate $P(E \cup F)$. (1)

(c) Find $P(E|F)$. (1)

[5]

30 marks

Total: 120 marks