NATIONAL SENIOR CERTIFICATE EXAMINATION
NOVEMBER 2008

ADVANCED MATHEMATICS

MARKING GUIDELINES

Time: 3 hours | 200 marks

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Module 1
Integration

QUESTION 1

1.1 $\int_1^2 3e^x\,dx = 3\int_1^2 e^x\,dx = 14{,}01$ (1)

1.2 $\int_1^{1,7} e^x\,dx + \int_{1,7}^2 [e^x + 2]\,dx$

$= \int_1^2 e^x\,dx + \int_{1,7}^2 2\,dx$ 1M

$= 4{,}67 + [2x]_{1,7}^2$ 1A for $2x$

$= 4{,}67 + 4 - 3{,}4$ 1M subst

$= 5{,}27$ 1A (4)

Max 3 if $m < A < M$ and uses $\frac{m+M}{2}$ [5]
Max 2 if stops $m < A < M$

QUESTION 2

2.1 Use 3 then double or 6 correct 1M

Area $= 100\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \frac{1}{2}f(x_3)\right]$ 1 if used

$= 100[50 + 125 + 200 + 162{,}5]$ 1M 1A

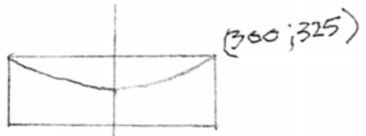
$= 53750$ mm^2 1A with mm^2 (5)

2.2 Vol $= \frac{53750}{1000000} \times 10$ 1A converts to meters

$= 0{,}05375 \times 10$ 1M mult by 10

$= 0{,}5375$ m^3 1A (3)

2.3



1M for $y = 325$

$$\text{Prism vol} = 0{,}6 \times 0{,}325 \times 10$$
$$= 1{,}95 \text{ m}^3$$ 1A

$$\therefore \text{Vol } H_2O = 1{,}95 - 0{,}5375$$
$$= 1{,}4125$$ 1CA (3)

2.4 2.2 is overestimate 1A (1)

[12]

QUESTION 3

$$\text{Area} = \int_0^k \sqrt{x}\, dx$$ 1M

$$= \left[\frac{2}{3} x^{3/2}\right]_0^k$$ 1A

$$= \frac{2}{3} k^{3/2}$$ 1A

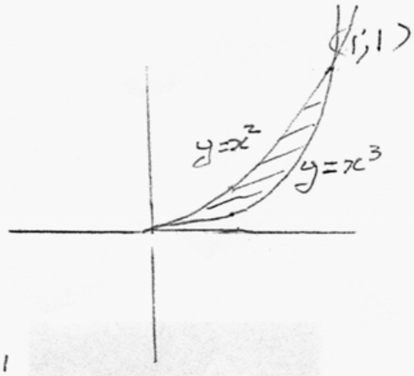
$$\therefore \frac{2}{3} k^{3/2} = 144$$ 1M

$$k^{3/2} = 216$$

$$k = 36$$ 1A

[5]

QUESTION 4



	1M for x^2 above x^3
	1M for $f^2 - g^2$
	1A for $(1;1)$
$\text{Vol} = \pi\int_0^1 y^{2/3}\,dy - \pi\int_0^1 y\,dy$	1A $y^{2/3}$ and y
$= \left[\frac{3}{5}y^{5/3}\right]_0^1 - \left[\frac{y^2}{2}\right]_0^1$ uses π ←	1A 1CA for integral
$= \frac{3\pi}{5} - \frac{\pi}{2}$	1M
$= \frac{\pi}{10}$ units^3	1A [8]

1M for some f^2 and g^2 as opposed to $(f-g)^2$ even if $f(x)$ instead of $f(y)$ etc.

Module 2.

1. 1.1 $\lim_{\theta \to 0} \frac{5\theta \cos\theta}{\sin 2\theta}$

$= \lim_{\theta \to 0} \frac{5\theta \cancel{\cos\theta}}{2\sin\theta \cancel{\cos\theta}}$ 1M

$= \lim_{\theta \to 0} \frac{5}{2} \frac{\theta}{\sin\theta}$ 1A (show $\frac{\theta}{\sin\theta}$)

$= \frac{5}{2} \cdot 1$ 1A (3)

1.2 $y = \frac{(1-x)^{1/2}}{(1+x)^{1/2}}$

$\frac{dy}{dx} = \frac{v(x).u'(x) - u(x)v'(x)}{[v(x)]^2}$ Max 2 if $\frac{uv' - vu'}{v^2}$

$= \frac{\sqrt{1+x} \cdot \frac{1}{2}(1-x)^{-1/2}(-1) - \sqrt{1-x} \cdot \frac{1}{2}(1+x)^{-1/2}}{1+x}$ 1A 1A 1A (3)

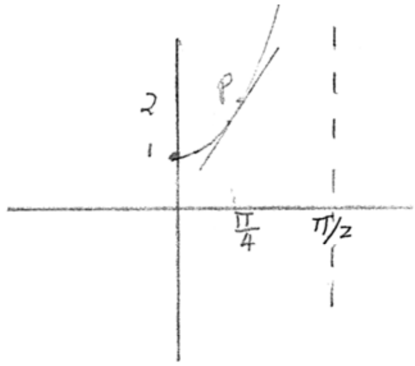
1.3 $f(\theta) = \sin(\cos\theta)$

$f'(\theta) = \cos(\cos\theta).(-\sin\theta)$ 1M chain rule 1A (2)

[8]

QUESTION 2

2.1



1A y int
1M Shape & asympt.

(2)

2.2. $y = \tan\theta + 1$

$\frac{dy}{dx} = \sec^2\theta$ 1A

Gradient at $\theta = \frac{\pi}{4}$ is $\frac{1}{\cos^2\frac{\pi}{4}} = 2.$ 1M

Tangent has equation $y = 2\theta + c$ 1M

But $P(\frac{\pi}{4};1)$ 1A

$\therefore 1 = 2(\frac{\pi}{4}) + c$

$1 = 1,57 + c$ 1M

$\therefore c = -0,57$

$\therefore$ Below origin (5)

[7]

QUESTION 3

3.1 $y = (4x - x^2)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(4x - x^2)^{-1/2}.(4 - 2x) = 0$ 1M 1A 1A(=0)

$\therefore \frac{4-2x}{2\sqrt{4x - x^2}} = 0$

$\therefore x = 2 \quad y = 2$ 1A 1CA (5)

Signtable or $\frac{d^2y}{dx^2}$ shows Max

$|x<| \; 2 \; |<x|$ 1M 1A (2)

QUESTION 4

4.1 Area of sector is $\frac{1}{2}r^2\theta$	1M
i.e. $A = \frac{1}{2}\,36\,\alpha$	
$= 18\alpha$	1A
area of Δ is $\frac{1}{2}.6.6.\sin\alpha$	
i.e. $18\sin\alpha$	1M
$\therefore$ Area segment $= 18\alpha - 18\sin\alpha$	(3)
4.2. $\frac{dA}{dt} = \frac{dA}{d\alpha}\cdot\frac{d\alpha}{dt}$	1A
$= (18 - 18\cos\alpha)(-\pi/3)$	1M 1A $\frac{d\alpha}{dt} = -\pi/3$
$= (18 - 18\cos\pi/3)(-\frac{\pi}{3})$	1M subst $\alpha = \frac{\pi}{3}$
$= (18 - 9)(-\frac{\pi}{3})$	
$= -3\pi.$	1A for answer -3π
	(5)
	[8]

Module 3

QUESTION 1

1.1

Midpoint $M(3;1;-3)$ — 1A

$BC = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}$ $\quad AM = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$ — 1M 1A

$\cos\alpha = \dfrac{MA \cdot BC}{|AM||BC|}$ — 1M if used, 1A

$= \dfrac{\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix}}{\left(\sqrt{4+1+16}\right)\left(\sqrt{4+16+4}\right)}$ — 1M, 1A $|AM||BC|$

$= \dfrac{-4+4+8}{\sqrt{21}\sqrt{24}}$

$\alpha = 69{,}12^\circ$ — 1A (8)

1.2. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$ — 1A 1A 1A (3)

[11]

QUESTION 2

2.1 On X axis, $y = 0$ & $z = 0$ — 1A

$\therefore 2x = 1$

$x = 1/2$ — 1A (2)

Can use parallel and a point (Max 1 if one of the other)

2.2. $x = 1+3s$; $y = -3+4s$; $z = 2+s$ — 1M

Subst in π: $2(1+3s) - (-3+4s) - 2(2+s)$ — 1M

$= 2+6s+3-4s-4-2s$

$= 1 = RHS.$ — 1A (4)

2.3 $d(C;\Pi)=\left\|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right\|$	1M if used
$=\frac{2(1)-1(-4)-2(-5)-1}{\sqrt{4+1+4}}$	1A subst
$=\frac{2+4+10-1}{3}$	
$=5$	1A
	(3)
	[9]

QUESTION 3

3.1 $l_1:\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\3\\5\end{pmatrix}+k\begin{pmatrix}2\\1\\-2\end{pmatrix}$	1M 1A
	(2)
3.2 $l_2:\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\-1\\2\end{pmatrix}+t\begin{pmatrix}6\\-2\\0\end{pmatrix}$	
A: If $x=-2$ on l_1, then $k=-2$	1A
$\therefore y=1$ and $z=9$	1M 1A for z
B: If $x=-2$ on l_2, then $t=-1$	1A
$\therefore y=1$ and $z=2$	1M must find y
Clearly $A(-2;1;9)$ and $B(-2;1;2)$ have same $x+y$ position but A is vertically above B.	1 explains only z differs
	(6)
3.3. Vertical sep is $9-2=7$	1M 1A (2)
	[10]

Matrices

QUESTION 1

1.1 $A = \begin{pmatrix} 3 & 3 \\ -2 & -1 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ 1A for B

$|A| = -9$ $|B| = 0$ 1A

$\therefore B$ is singular

as determinant $= 0$ 1 [3]

1.2

1.2.1 $P^2 = P - I$

$\therefore P^{-1}PP = P^{-1}P - P^{-1}$ 1M

$P = I - P^{-1}$ 1A

$\therefore P^{-1} = I - P$ 1A (3)

1.2.2 $P^2 = P - I$

$PP^2 = PP - PI$ 1M

$P^3 = P^2 - P$

$= P - I - P$ 1A

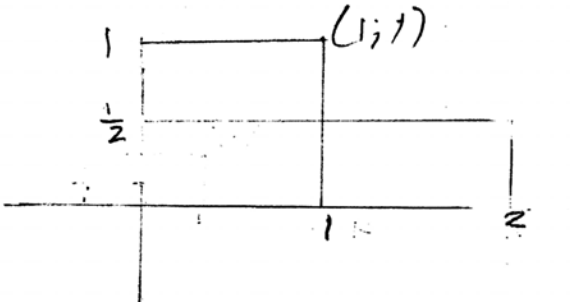
$= -I$ (2)

$\therefore P^3 + I = 0$

[8]

QUESTION 2

2.1.1



1A

$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ 1M (2)

2.1.2 Area square $= 1 \times 1 = 1$

Area rect $= 2 \times \frac{1}{2}$ 1M 1A (2)

$= 1$

2.2

2.2.1 Grad $= \tan\alpha = \sqrt{3}$

$\therefore \alpha = 60°$ 1A (1)

2.2.2 $P = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ 1A if used

$= \begin{pmatrix} \cos -60° & -\sin -60° \\ \sin -60° & \cos -60° \end{pmatrix}$

$= \begin{pmatrix} 0,5 & \sqrt{3}/2 \\ -\frac{\sqrt{3}}{2} & 0,5 \end{pmatrix}$ 1A either way But must be $-60°$; $300°$; or $120°$ (2)

2.2.3 $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (1)

2.2.4

Rotate $-60°$

$\begin{pmatrix} 0,5 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0,5 \end{pmatrix}\begin{pmatrix} -1 \\ 2\sqrt{3} \end{pmatrix} = \begin{pmatrix} 5/2 \\ \frac{3\sqrt{3}}{2} \end{pmatrix}$ 1M premultiply by any 2.2.2

then reflect $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{5}{2} \\ \frac{3\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} 5/2 \\ -\frac{3\sqrt{3}}{2} \end{pmatrix}$ 1M uses his 2.2.3

then rotate $60°$

$\begin{pmatrix} 0,5 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0,5 \end{pmatrix}\begin{pmatrix} \frac{5}{2} \\ -\frac{3\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{7}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ 1M opp. direction 1A

(4)

Can use $120°$ or $300°$ follow same scheme + method [12]

QUESTION 3

3.1

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 4 & 5 & 6 & 24 \\ 2 & 7 & 12 & 30 \end{array}\right) \xrightarrow{R_1;\ R_2-4R_1;\ R_3-2R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -3 & -6 & -12 \\ 0 & 3 & 6 & 12 \end{array}\right) \xrightarrow{R_3+R_2}$$

2M

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -3 & -6 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1;\ R_2 \div -3;\ R_3}$$

1A

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

or $z = t$, $x = 1+t$, $y = 4-2t$ etc

Let $y = t$, then $t + 2z = 4$ 1M

$\therefore z = \frac{4-t}{2}$ 1A

$x + 2t + \frac{3(4-t)}{2} = 9$

$2x + 4t + 12 - 3t = 18$ $\therefore x = \frac{6-t}{2}$ 1A (6)

eg $(1,4,0)$ or "infinitely many" then Max $\frac{4}{6}$

3.2 Now t must be > 0

Also $4 - t > 0$ i.e. $t < 4$

$6 - t > 0$ i.e. $t < 6$

1M Tries to get integers

2M Clear argument

Try $t = 1$ No, x & $y \notin Z^+$

$t = 2$ $x = 2, y = 2, z = 1$ 1A

$t = 3$ No, x & $y \notin Z^+$

(4)

[10]

Module 5

QUESTION 1

1.1 Trial + error $N = 2$ 1M 1A

or $\frac{40}{2^x} - x^2 = 0$

$x = 1 \quad f(x) > 0$

$x = 2 \quad f(x) > 0$

$x = 3 \quad f(x) < 0$ (2)

1.2. $\alpha = 1{,}132$ because $|-0{,}00214| < 0{,}003$ 2A (2)

[4]

QUESTION 2

2.1 $x_{r+1} = 3 + \frac{1}{x_r}$

$x_1 = 3 + \frac{1}{3{,}5} = 3{,}2857143$ 1M [-1 if wrong dec as per and in 4.1]

$x_2 = 3 + \frac{1}{3{,}28\ldots} = 3{,}3043478$ 1A

$x_3 = 3 + \frac{1}{3{,}304\ldots} = 3{,}3026316$

$= 3{,}30263$ 1A (3)

2.2 $x = 3 + \frac{1}{x}$ 1M

$x^2 = 3x + 1$

$x^2 - 3x - 1 = 0$

$\therefore P(x) = x^2 - 3x - 1$ 1A (2)

[5]

QUESTION 3

3.1

0	3,5			
		$-0,1$ ✓A		
4	3,1		$\frac{-0,15}{6} = -0,025$ ✓M	
		$-0,25$ ✓A		
6	2,6		$\frac{-0,1}{2} = -0,05$	
		$-0,35$		
8	1,9			

$h = f(x_0) + (x - x_0)f[x_0; x_1] + (x-x_0)(x-x_1)f[x_0;x_1;x_2]$ 1A ✓ used

$= 3,5 + (5 - 0)(-0,1) + (5)(5-4)(-0,025)$ 1M subst

$= 3,5 + -0,5 + 0,125$

$= 3,125$ 1M (6)

3.2

3,1	4	$\frac{2}{-0,5} = -4$
2,6	6	

1M 1A

$x = f(h_0) + (h - h_0)f[h_0; h_1]$

$= 4 + (2,8 - 3,1)(-4)$ 1M

$= 4 + (-0,3)(-4)$

$= 4 + 1,2$

$= 5,2$ 1A (4)

[10]

QUESTION 4

4.1 $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ 1A if used

$\therefore x_{n+1} = x_n - \dfrac{x_n^3 - 3x_n - 4}{3x_n^2 - 3}$ 1A $f'(x_n)$

$= \dfrac{3x_n^3 - 3x_n - x_n^3 + 3x_n + 4}{3x_n^2 - 3}$

$= \dfrac{2x_n^3 + 4}{3x_n^2 - 3}$ 1M 1A

$x_0 = 2,5 \quad \therefore x_1 = \dfrac{2(2,5)^3 + 4}{3(2,5)^2 - 3}$

$= 2,2380952$ 1M

$x_2 = 2,1968146$ 1A

$x_3 = 2,1958239$

$x_4 = 2,1958233$

$= 2,196$ 1CA (7)

4.2

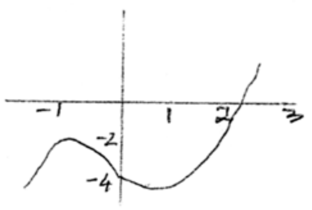
$3x^2 - 3 = 0$ at stat pts

$x = \pm 1$ 1A

$x<$	-1	0	1	$<x$
$f'(x)$ +	0	−	0	+

(or use f'') 1M

Max at $(-1;-2)$
Min at $(1;-6$ 1A both



Max 2 if sketch only unless at least 4 points shown. 1M (4)

[11]

Module 6
Logic + Boolean Algebra

QUESTION 1

1.1

a	b	$a+\bar{b}$	$(a+\bar{b}).b$	$\Rightarrow$	a
T	T	T	T	T	T
T	F	T	F	T	T
F	T	F	F	T	F
F	F	T ✓	F ✓	T ✓	F ✓

$\therefore$ all T ✓ 5A (5)

1.2 1.2.1 $y \Rightarrow x$ 1A

1.2.2 $y \Rightarrow x$ 1A

1.2.3 $-y \Rightarrow -x$ 1A (3)

1.3 False 1A
If $x<0$ then no y so that $y^2 = x$
since $y^2 \geq 0$ for all y 1A (2)

[10]

QUESTION 2

Suppose that

$$\frac{p}{2q} + \frac{2q}{p} < 2$$ 1M

$\times$ by $2pq > 0$ to get 1M

$$p^2 + 4q^2 < 4pq$$ 1A

$$\therefore \quad p^2 - 4pq + 4q^2 < 0$$

$$\therefore (p-2q)^2 < 0$$ 1A

which is impossible since 1
all squares are non-negative
$\therefore$ Supposition is wrong 1 [6]

QUESTION 3

Step 1 Let $n = 1$

$\therefore LHS = 1 = RHS$ 1A

i.e proposition true for $n = 1$ 1

Step 2

assume prop. true for $n = k$

ie

$1 + \frac{6}{7} + \cdots \left(\frac{6}{7}\right)^{k-1} = 7\left(1 - \left(\frac{6}{7}\right)^k\right)$ 1M

$\therefore 1 + \frac{6}{7} + \cdots \left(\frac{6}{7}\right)^{k-1} + \left(\frac{6}{7}\right)^k = 7\left(1 - \left(\frac{6}{7}\right)^k\right) + \left(\frac{6}{7}\right)^k$ 1A add $\left(\frac{6}{7}\right)^k$ to both

$= 7 - 7\left(\frac{6}{7}\right)^k + \left(\frac{6}{7}\right)^k$ 1

$= 7 - 6\left(\frac{6}{7}\right)^k$

$= 7\left(1 - \left(\frac{6}{7}\right)\left(\frac{6}{7}\right)^k\right)$ 1

$= 7\left(1 - \left(\frac{6}{7}\right)^{k+1}\right)$ 1A

Hence if prop true for $n = k$, it is true for $n = k + 1$ 1

$\therefore$ by M.I it is true for $n \in N$ [8]

QUESTION 4

$(A.\bar{B}) + B + (\bar{A}.B)$ 1A 1A

$= (A + B).(\bar{B} + B) + (\bar{A}.B)$ 1M

$= (A + B).\ I + \bar{A}.B$ 1A

$= A + B + \bar{A}B$

$= A + (I + \bar{A}).B$ 1M

$= A + I.B$

$= A + B$ 1A [6]

Module 7
Prob, perms. + combs.

QUESTION 1

1.1 G G G G Seat girls 4! ways 1A

Seat boys in the gaps ~~6~~ 5! ways 1A

$\therefore$ Total # ways $= 4!\,5!$

$= 2880$ 1CA (3)

1.2

Total possibilities − # together

$= 5! - 2.4!$ 1A 1A

$= (5-2)4!$

$= 3.4!$

$= 72$ 1A (3)

1.3

ways $= (2!\,2!\,3!\,4!)4!$ 2A 1A

$= 13824$ 1A (4)

1.4 3 digit

start with 4 : 1.2.2 = 4

start with 5 : 1.2.1 = 2

4 digit

Start with 2 : 1.2.1.2 = 4

Start with 3 : 1.2.1.1 = 2

" " 4 : 1.2.1.2 = 4

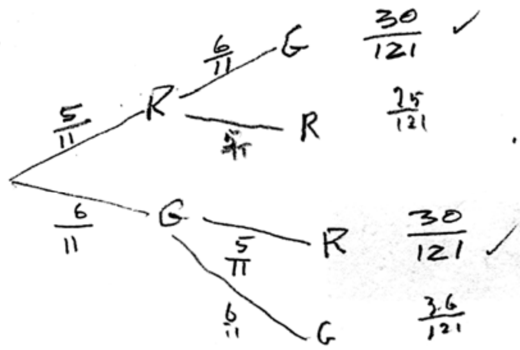
" " 5 : 1.2.1.1 = 2

Finally 18 (5)

[15]

QUESTION 2

2.1



$\therefore P = \frac{60}{121}$ ✓ | 2M 1A

0,4987

(3)

2.2

2.2.1 (i) $P(E \cap F) = 0{,}1$ and not $0{,}18$ | 1A

$P(E).P(F) = 0{,}18$

ii) $P(E \cap F) = 0{,}1 \neq 0$

$\therefore E \cap F \neq \phi$ | 1A (2)

2.2.2 i) $P(\bar{E}) = 1 - 0{,}3$

$= 0{,}7$ | 1A

ii) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ | 1M

$= 0{,}3 + 0{,}6 - 0{,}1$

$= 0{,}8$ | 1A

iii) $P(E|F) = \frac{P(E \cap F)}{P(F)}$ | 1M

$= \frac{0{,}1}{0{,}6}$

$= \frac{1}{6}$ | 1A

iv) $P(\bar{E}|F) = \frac{5}{6}$ | 1A (6)

2.3 Let X be the number of defective drawn in the sample of 5.

Clearly $P(\text{defective}) = \frac{1}{10}$ — 1A

$P(X=0) = \binom{5}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^5 = 0,59$ — 1M

$P(X=1) = \binom{5}{1}\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^4 = \cancel{0,07}$ $0,33$ — 1M

$\therefore P(X \leq 1) =$ — 1A

$0,92$

(4)

[15]

Total: 200 marks