GRADE 12 EXAMINATION
NOVEMBER 2008

ADVANCED MATHEMATICS

Time: 2½ hours | 120 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 13 pages. Please check that your question paper is complete.
2. Answer ALL the questions set on FOUR of the modules you have studied.
3. Non-programmable calculators may be used, unless otherwise indicated.
4. All necessary calculations must be clearly shown and writing should be legible.
5. Diagrams have not been drawn to scale.

MODULE 1 INTEGRATION

QUESTION 1

Given that $\int_1^2 e^x dx = 4,67$ find

1.1 $\int_1^2 3e^x dx$ (1)

1.2 $\int_1^{1,7} e^x dx + \int_{1,7}^2 [e^x + 2]dx$ (4)

[5]

QUESTION 2

Figure (i) shows the curve $y = \frac{1}{400}x^2 + 100$ for $-300 \le x \le 300$
The units involved are millimeters.
The shaded region in figure (i) is the cross-section of a concrete drainage channel as shown in figure (ii).

Figure (i)

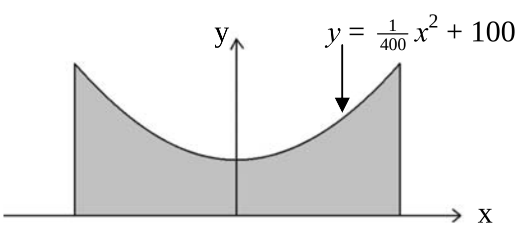
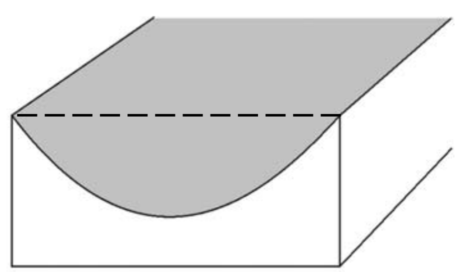


Figure (ii)



2.1 Use the Trapezium Rule with 6 subintervals to find the approximate area of the cross-section of the concrete channel. (Give the correct units) (5)

2.2 Hence, find the approximate volume of concrete in a 10 meter length of channel. (Give your answer in cubic meters) (3)

2.3 Approximately how many cubic meters of water are contained in a 10 meter length of channel when it is full? (i.e. the shaded volume in Figure (ii)) (3)

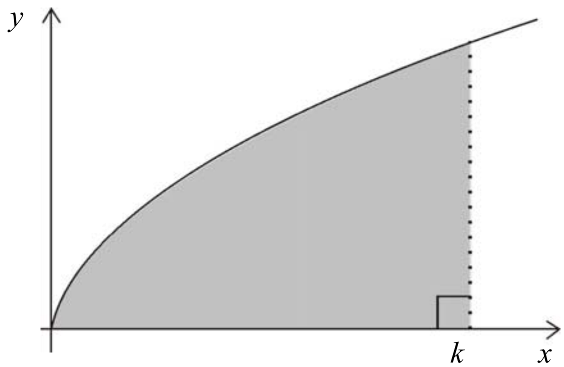
2.4 Of the answers in 2.2 and 2.3, which is an overestimate? (1)

[12]

QUESTION 3

The region bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = k$ is shown in the sketch below.

Find k if the area of the shaded region is 144 units^2.



[5]

QUESTION 4

An ornamental bowl matches the solid formed by rotating the area between

$$y = x^2,\ x \geq 0 \quad \text{and} \quad y = x^3,\ x \geq 0,$$ through 360° about the y–axis.

Calculate the volume of glass in the bowl.

[8]

30 marks

MODULE 2 DIFFERENTIATION

QUESTION 1

1.1 Find the value of $\lim_{\theta \to 0} \frac{5\theta \cos\theta}{\sin 2\theta}$, showing clearly how you arrived at your answer. (3)

1.2 Find $\frac{dy}{dx}$ if $y = \frac{\sqrt{1-x}}{\sqrt{1+x}}$. (Leave your answer unsimplified) (3)

1.3 Differentiate $f(\theta) = \sin(\cos\theta)$ (2)

[8]

QUESTION 2

2.1 Sketch the graph of $y = \tan\theta + 1$ for $0 \le \theta < \frac{\pi}{2}$. (2)

2.2 Investigate whether the tangent to the curve $y = \tan\theta + 1$ at $\theta = \frac{\pi}{4}$ cuts the y–axis above or below the origin. **(Show your working)** (5)

[7]

QUESTION 3

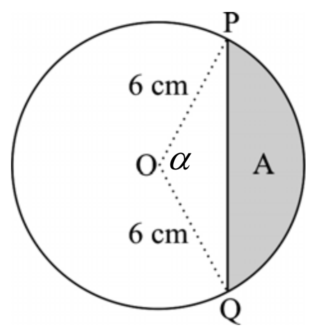
3.1 Find the co-ordinates of the stationary point of the curve whose equation is $y = \sqrt{4x - x^2}$. (5)

3.2 Investigate whether this stationary point is a maximum or a minimum point. (2)

[7]

QUESTION 4

Refer to the circle below with centre O and radius 6 cm.
The chord PQ subtends angle α at centre O.



4.1 Show that the shaded area $A = 18\alpha - 18 \sin \alpha$. (3)

4.2 If α decreases at a rate of $\frac{\pi}{3}$ radians per minute, find the rate of change of area A when $\alpha = \frac{\pi}{3}$. (5)

[8]

30 marks

MODULE 3 VECTORS

QUESTION 1

Points A, B and C have co-ordinates (1; 2; 1); (2; –1; –4) and (4; 3; –2) respectively.

1.1 Calculate the acute angle between side BC and median AD of triangle ABC. (8)

1.2 Find a vector equation of the plane through A, B and C. (3)

[11]

QUESTION 2

The equation of plane π is $2x - y - 2z = 1$.

2.1 Find the co-ordinates of the point where plane π intersects the x-axis. (2)

2.2 Investigate whether or not the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ is part of plane π. (4)

2.3 Calculate the radius of a sphere with centre (1; –4; –5) if plane π is a tangent plane to the sphere. (3)

[9]

QUESTION 3

Given P (2; 3; 5) and Q (4; 4; 3),

3.1 Write down a vector equation of the line ℓ_1 through P and Q. (2)

3.2 A second line ℓ_2 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix}$

A is a point on ℓ_1 where $x = -2$.
Show that A is directly above a particular point (call it point B) on ℓ_2. (6)

3.3 Find the vertical separation AB of the lines ℓ_1 and ℓ_2, i.e. the vertical distance between the lines where ℓ_1 passes over ℓ_2. (2)

[10]

30 marks

MODULE 4 MATRICES

QUESTION 1

1.1 Given $A = \begin{pmatrix} 3 & 3 \\ -2 & -1 \end{pmatrix}$

and given a 2×2 matrix $B = b_{i\,j}$ for which

$$b_{i\,j} = (j)^{i}, \text{ for } i = j$$

and $$b_{i\,j} = i \times j \text{ for } i \neq j$$

Which of A and B is not invertible? Explain. (3)

1.2 P is a 3×3 matrix such that $P^2 = P - I$, where I is the 3×3 unit matrix.

1.2.1 Find P^{-1} in terms of P and I. (3)

1.2.2 Now show that $P^3 + I = 0$, where 0 is the 3×3 zero matrix. (2)

[8]

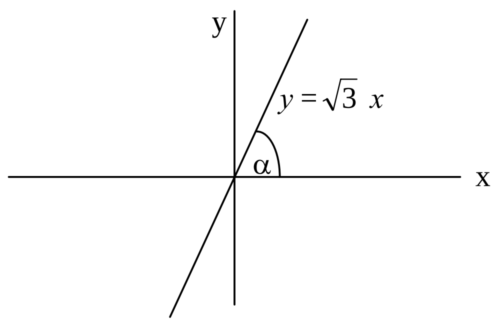
QUESTION 2

2.1 A transformation T is represented by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

2.1.1 Draw a diagram showing a unit square and its image under T. (2)

2.1.2 Show that the area of the image of the square under transformation T is the same as the area of the square. (2)

2.2 The sketch below shows the graph of $y = \sqrt{3}x$.



2.2.1 Calculate the size of angle α. (1)

2.2.2 Find the matrix P that rotates the line $y = \sqrt{3}x$ onto the positive x axis. (2)

2.2.3 Write down the matrix A that reflects any point (a;b) in the x axis. (1)

2.2.4 Now find the reflection of the point $(-1;\ 2\sqrt{3})$ in the line $y = \sqrt{3}x$. (4)

[12]

QUESTION 3

3.1 Use row reduction to find all the solutions of the system of equations

$$\begin{aligned} x + 2y + 3z &= 9 \\ 4x + 5y + 6z &= 24 \\ 2x + 7y + 12z &= 30 \end{aligned}$$

(6)

3.2 If it is further given that x, y and z are all positive integers, show that there is only one solution to the system of equations.
Write down this solution. (4)

[10]

30 marks

MODULE 5 NUMERICAL METHODS

QUESTION 1

1.1 Find the positive integer N such that $\frac{40}{2^x} = x^2$ has a root between N and N+1. (2)

1.2 A numerical method was used to solve the equation $p(x) = 0$

The following table gives values of $p(x)$ for various x values:

x	1,1	1,2	1,13	1,14	1,132	1,134
$p(x)$	–0,269	0,538	–0,037	0,04154	–0,00214	0,01

Give the value of α if $x = \alpha$ is an approximate solution to $p(x) = 0$, such that

$$|p(\alpha)| < 3 \times 10^{-3} .$$ (2)

[4]

QUESTION 2

$f(x) = 0$ has a root $x = \beta$ such that $3 < \beta < 4$.

The iterative formula

$$x_{r+1} = 3 + \frac{1}{x_r} \quad \text{for all } r \in \text{N},$$

is used to calculate a sequence of approximations to β.

2.1 Using $x_0 = 3{,}5$ find x_1; x_2 , and x_3, correct to 5 decimal places. (3)

2.2 It is also given that the sequence x_1; x_2; x_3 … converges to a limit.
Find polynomial $f(x)$. (2)

[5]

QUESTION 3

An irrigation system squirts a stream of water horizontally from a nozzle set on an irrigation platform 3,5 meters above ground.
The height h of the stream at various distances x along level ground appears in the table below:

Distance x meters	$x = 0$	$x = 4$	$x = 6$	$x = 8$
Height h meters	3,5	3,1	2,6	1,9

3.1 Use quadratic interpolation to estimate the height of the stream at a distance 5 m from the foot of the irrigation platform. (6)

3.2 Calculate the distance x for the stream to reach a height of approximately 2,8 meters using linear interpolation. (4)

[10]

QUESTION 4

The equation $x^3 - 3x - 4 = 0$ has a root in the interval [2; 3].

4.1 Use the Newton-Raphson method to find this root correct to three places of decimals. (7)

4.2 Explain clearly why this equation has no other roots. (4)

[11]

30 marks

MODULE 6 LOGIC AND BOOLEAN ALGEBRA

QUESTION 1

1.1 Show by means of a truth-table, or otherwise, that the statement

$$[(a + \overline{b}).b] \Rightarrow a$$

is a tautology. (5)

1.2 Let x mean 'the days are growing longer'; and y mean 'summer is approaching'.

Use x and y as defined above to write the following statements in symbolic form.

1.2.1 Approaching summer is sufficient for longer days.

1.2.2 The days grow longer because summer is approaching.

1.2.3 A necessary condition for shorter days is that summer be receding. (3)

1.3 Explain whether the following statement is true or false:

$$(\forall x \in \mathrm{R})(\exists y \in \mathrm{R}) : (y^2 = x)$$ (2)

[10]

QUESTION 2

Prove by the method of contradiction that

$$\frac{p}{2q} + \frac{2q}{p} \geq 2$$

if p and q are positive real numbers. **[6]**

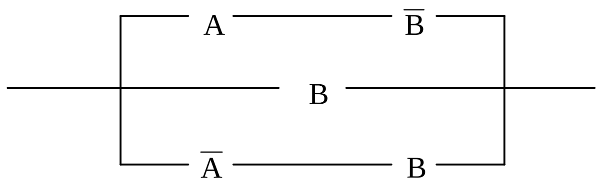
QUESTION 3

Prove by mathematical induction that $\forall n \in \mathrm{N}$,

$$1 + (\tfrac{6}{7}) + (\tfrac{6}{7})^2 + \ldots\ldots + (\tfrac{6}{7})^{n-1} = 7(1 - (\tfrac{6}{7})^n)$$ **[8]**

QUESTION 4

The sketch shows part of an electric network. Write down and simplify the Boolean expression for this network.



[6]

30 marks

MODULE 7 PROBABILITY, PERMUTATIONS AND COMBINATIONS

QUESTION 1

1.1 In how many ways can 5 boys and 4 girls be seated in a row so that boys and girls are placed alternately? (3)

1.2 In how many ways can 6 people be arranged at a round table so that Percival and Stan do not sit together? (3)

1.3 Brother Benedict, a Medieval monk, had in his cell

2 books on geometry
2 books on music
3 books on arithmetic
4 books on astronomy.

He kept his books on a single shelf so that books on the same subject were together. In how many ways could he have arranged his books on his bookshelf? (4)

1.4 How many different odd numbers greater than 400 can be made from the digits 2;3;4 and 5 if each digit is used once only? (5)

[15]

QUESTION 2

2.1 A bag contains 5 red and 6 green marbles.
Two marbles are drawn at random from the bag.
What is the probability of drawing marbles of different colour if the first marble is replaced before the second marble is drawn? (3)

2.2 E and F are events of a sample space S.
It is given that $P(E) = 0{,}3$; $P(F) = 0{,}6$ and $P(E \cap F) = 0{,}1$.

2.2.1 How do we know that E and F are not (i) independent events?
(ii) mutually exclusive events? (2)

2.2.2 Find:

(i) $P(\overline{E})$
(ii) $P(E \cup F)$
(iii) $P(E \mid F)$
(iv) $P(\overline{E} \mid F)$ (6)

2.3 If 10% of the light bulbs produced by a factory are defective, what is the probability that out of 5 bulbs chosen at random, fewer than 2 will be defective? (4)

[15]

30 marks

Total: 120 marks